Stability and asymptotic behavior of periodic traveling wave solutions of viscous conservation laws in several dimensions*

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Abstract. Under natural spectral stability assumptions motivated by previous investigations of the associated spectral stability problem, we determine sharp L^p estimates on the linearized solution operator about a multidimensional planar periodic wave of a system of conservation laws with viscosity, yielding linearized $L^1 \cap L^p \to L^p$ stability for all $p \geq 2$ and dimensions $d \geq 1$ and nonlinear $L^1 \cap H^s \to L^p \cap H^s$ stability and L^2-asymptotic behavior for $p \geq 2$ and $d \geq 3$. The behavior can in general be rather complicated, involving both convective (i.e., wave-like) and diffusive effects.

1 Introduction

Nonclassical viscous conservation laws arising in multiphase fluid and solid mechanics exhibit a rich variety of traveling wave phenomena, including homoclinic (pulse-type) and periodic solutions along with the standard heteroclinic (shock, or front-type) solutions. Here, we investigate stability of periodic traveling waves: specifically, sufficient conditions for stability of the wave. Our main result is to establish L^p bounds on the solution operator for the linearized evolution equations, provided that there holds an appropriate spectral condition on the linearized operator about the wave. An immediate consequence is that, under mild nondegeneracy assumptions motivated by the low-frequency spectral analysis of [OZ3], *strong spectral stability in the sense of Schneider [S1,S2,S3] implies linearized and nonlinear $L^1 \cap H^s \to L^p \cap H^s$ asymptotic stability*, for all $p \geq 2$ and dimensions $d \geq 3$.

The one-dimensional study on spectral stability of spatially periodic traveling waves of systems of viscous conservation laws was carried out by Oh & Zumbrun [OZ1] in the "quasi-Hamiltonian" case that the traveling-wave equation possesses an integral of motion, and in the general case by Serre [Se1]. An important contribution of Serre was to point out a larger connection between the linearized dispersion relation (the function $\lambda(\xi)$ relating spectra to wave number of the linearized operator about the wave) near zero and the homogenized system obtained by slow modulation, or WKB, approximation, from which the various stability results of [OZ1], [Se1] may then be deduced. In [OZ3], we extended this important observation of Serre, relating the linearized dispersion relation near zero to a multi-dimensional version of the homogenized system. As an immediate corollary, similarly

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as in [OZ1], [Se1] in the one-dimensional case, this yielded as a necessary condition for multi-dimensional stability the hyperbolicity of the multi-dimensional homogenized system.

As noted in [OZ3], this relation is also the first step in the study of stability and asymptotic behavior. In this article, we use the description of low-frequency spectrum carried out in [OZ3] to obtain nonlinear stability and asymptotic behavior in dimensions $d \geq 3$ by a modification of the Bloch decomposition arguments introduced by Schneider in [S1,S2,S3]; see Theorems 1, 2, and 3. Here, the main new difficulty is the fact that spectra $\lambda(\xi)$ bifurcating from the translational zero eigenvalue at $\xi = 0$ are not smooth at the origin, a standard feature of hyperbolic–parabolic systems in multi-dimensions [ZS,Z1]. This is not only a technical issue, but reflects quite different behaviors in the present vs. previously considered cases. Whereas asymptotic behavior in [S1,S2,S3] was purely diffusive, corresponding to a Gaussian kernel, the behavior here is convective–diffusive, corresponding asymptotically to a convection–diffusion wave in the sense of [HoZ1]. Stability and behavior in dimensions one and two remain interesting open problems.

2 Preliminaries

Consider a system of conservation laws

$$u_t + \sum_j f^j(u)_{x_j} = \sum_{j,\,k} \Big(B^{jk}(u) u_{x_k}\Big)_{x_j}, \tag{2.1}$$

$u \in \mathcal{U}(\text{open}) \in \mathbb{R}^n$, $f^j \in \mathbb{R}^n$, $B^{jk} \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^d$, and a periodic traveling wave solution

$$u = \bar{u}(x\cdot\nu - st), \tag{2.2}$$

of period X, satisfying the traveling-wave ordinary differential equation (ODE)

$$(\sum_{j,k} \nu_j \nu_k B^{jk}(\bar{u})\bar{u}')' = (\sum_j \nu_j f^j(\bar{u}))' - s\bar{u}' \tag{2.3}$$

with initial conditions

$$\bar{u}(0) = \bar{u}(X) =: u_0.$$

Integrating (2.3), we reduce to a first-order profile equation

$$\sum_{j,k} \nu_j \nu_k B^{jk}(\bar{u})\bar{u}' = \sum_j \nu_j f^j(\bar{u}) - s\bar{u} - q \tag{2.4}$$

encoding the conservative structure of the equations, where q is a constant of motion.

We here briefly review the generic assumptions made in [OZ3,Se1]. Given

$$(a, s, \nu, q) \in \mathcal{U} \times \mathbb{R} \times S^{d-1} \times \mathbb{R}^n,$$

(2.4) admits a unique local solution $u(y; a, s, \nu, q)$ such that $u(0; a, s, \nu, q) = a$. Denote by X the period, $\omega := 1/X$ the frequency and M and F^j the averages over the period:

$$M := \frac{1}{X}\int_0^X u(y)dy, \quad F^j := \frac{1}{X}\int_0^X \Big(f^j(u) - \sum_{k=1}^d B^{jk}(u)\omega\nu_k\partial_y u\Big)dy$$

when u is a periodic solution of (2.4). Since these quantities are translation invariant, we consider the set P of periodic functions u that are solutions of (2.4) for some triple (s, ν, q), and construct the quotient set $\mathcal{P} := P/\mathcal{R}$ under the relation

$$(u\,\mathcal{R}\,v) \iff (\exists h \in \mathbb{R};\ v = u(\cdot - h)).$$

We thus have class functions:

$$X = X(\dot{u}),\ \omega = \Omega(\dot{u}),\ s = S(\dot{u}),\ \nu = N(\dot{u}),\ q = Q(\dot{u}),\ M = M(\dot{u}),\ F^j = F^j(\dot{u}), \quad (2.5)$$

where $\dot{u}$ is the equivalence class of translates of different periodic functions. Note that $\bar{u}$ is a nonconstant periodic solution. Without loss of generality, assume $S(\bar{u}) = 0$ and $N(\bar{u}) = e_1$, so that (2.4) takes the form

$$B^{11}(\bar{u})\bar{u}' = f^1(\bar{u}) - \bar{q}$$

for $\bar{q} = Q(\bar{u})$. Letting $\bar{X} = X(\bar{u})$ and $\bar{a} = \bar{u}(0) = u_0$, the map

$$(y, a, s, \nu, q) \mapsto u(y; a, s, \nu, q) - a$$

is smooth and well-defined in a neighborhood of $(\bar{X}; \bar{a}, 0, e_1, \bar{q})$, and it vanishes at this special point. Here and elsewhere, e_j denotes the jth standard Euclidean basis element. We assume:

(H0) f^j, $B^{jk} \in C^k$, $k \geq [d/2] + 1$.

(H1) Re $\sigma(\sum_{jk} \nu_j\nu_k B^{jk}) \geq \theta > 0$ for all $\nu \in S^{d-1}$.

(H2) The map $H : \mathbb{R} \times \mathcal{U} \times \mathbb{R} \times S^{d-1} \times \mathbb{R}^n \to \mathbb{R}^n$ taking $(X; a, s, \nu, q) \mapsto u(X; a, s, \nu, q) - a$ is a submersion at point $(\bar{X}; \bar{a}, 0, e_1, \bar{q})$.

3 The Evans function

Without loss of generality taking $S(\bar{u}) = 0$, $N(\bar{u}) = e_1$, $\bar{u} = \bar{u}(x_1)$ represents a stationary solution. Linearizing (2.1) about $\bar{u}(\cdot)$, we obtain

$$v_t = Lv := \sum (B^{jk}v_{x_k})_{x_j} - \sum (A^j v)_{x_j}, \quad (3.1)$$

where coefficients

$$B^{jk} := B^{jk}(\bar{u}), \quad A^j v := Df^j(\bar{u})v - (DB^{j1}(\bar{u})v)\bar{u}_{x_1} \quad (3.2)$$

are now periodic functions of x_1.

Taking the Fourier transform in the transverse coordinate $\tilde{x} = (x_2, \cdots, x_d)$, we obtain

$$\begin{aligned} \hat{v}_t = L_{\tilde{\xi}}\hat{v} = (B^{11}\hat{v}_{x_1})_{x_1} - (A^1\hat{v})_{x_1} + i(\sum_{j\neq 1} B^{j1}\xi_j)\hat{v}_{x_1} \\ + i(\sum_{k\neq 1} B^{1k}\xi_k\hat{v})_{x_1} - i\sum_{j\neq 1} A^j\xi_j\hat{v} - \sum_{j\neq 1, k\neq 1} B^{jk}\xi_k\xi_j\hat{v}, \end{aligned} \tag{3.3}$$

where $\tilde{\xi} = (\xi_2, \cdots, \xi_d)$ is the transverse frequency vector. The Laplace transform in time t leads us to study the family of eigenvalue equations

$$\begin{aligned} 0 = (L_{\tilde{\xi}} - \lambda)w = (B^{11}w')' - (A^1w)' + i\sum_{j\neq 1} B^{j1}\xi_j w' + i(\sum_{k\neq 1} B^{1k}\xi_k w)' \\ - i\sum_{j\neq 1} A^j\xi_j w - \sum_{j\neq 1, k\neq 1} B^{jk}\xi_k\xi_j w - \lambda w, \end{aligned} \tag{3.4}$$

associated with operators $L_{\tilde{\xi}}$ and frequency $\lambda \in \mathbb{C}$, where '$'$' denotes $\partial/\partial x_1$. Clearly, a necessary condition for stability of (2.1) is that (3.4) have no L^2 solutions w for $\tilde{\xi} \in \mathbb{R}^{d-1}$ and Re $\lambda > 0$. For solutions of (3.4) correspond to normal modes $v(x,t) = \mathrm{e}^{\lambda t}\mathrm{e}^{i\tilde{\xi}\cdot\tilde{x}}w(x_1)$ of (3.1).

Multidimensional stability concerns primarily the behavior of the perturbation of the top eigenvalue $\lambda = 0$ under small perturbations in $\tilde{\xi}$. To study this behavior, we use Floquet's theory and an Evans function that not only depends on λ but also on ξ_1 (which corresponds to the phase shift) and $\tilde{\xi}$ [G,OZ3]. To define the Evans function, we choose a basis $\{w^1(x_1, \tilde{\xi}, \lambda), \ldots, w^{2n}(x_1, \tilde{\xi}, \lambda)\}$ of the kernel of $L_{\tilde{\xi}} - \lambda$, which is analytic in $(\tilde{\xi}, \lambda)$ and is real when λ is real, for details see [OZ1,Se1]. Now we can define the Evans function by

$$D(\lambda, \xi_1, \tilde{\xi}) := \left| \begin{matrix} w^l(X, \tilde{\xi}, \lambda) - \mathrm{e}^{iX\xi_1}w^l(0, \tilde{\xi}, \lambda) \\ (w^l)'(X, \tilde{\xi}, \lambda) - \mathrm{e}^{iX\xi_1}(w^l)'(0, \tilde{\xi}, \lambda) \end{matrix} \right|_{1\le l\le 2n} \tag{3.5}$$

where $\xi_1 \in \mathbb{R}$. We remark that D is analytic everywhere, with associated analytic eigenfunction w^l for $1 \le l \le 2n$. A point λ is in the spectrum of $L_{\tilde{\xi}}$ if and only if $D(\lambda, \xi) = 0$ with $\xi = (\xi_1, \tilde{\xi})$.

Example 31. In the constant-coefficient case, $D(\lambda, \xi) = \prod_{l=1}^{2n}(\mathrm{e}^{\mu_l(\lambda,\tilde{\xi})X} - \mathrm{e}^{i\xi_1 X})$, where $\mu_l, l = 1, \ldots, 2n$, denote the roots of the characteristic equation

$$\begin{aligned} \Big(\mu^2 B^{11} + \mu(-A^1 + i\sum_{j\neq 1} B^{j1}\xi_j + i\sum_{k\neq 1} B^{1k}\xi_k) \\ - (i\sum_{j\neq 1} A^j\xi_j + \sum_{j\neq 1, k\neq 1} B^{jk}\xi_k\xi_j + \lambda I)\Big)\bar{w} = 0, \end{aligned} \tag{3.6}$$

with $w = \mathrm{e}^{\mu x_1}\bar{w}$. The zero set of D consists of all λ and ξ_1 such that $\mu_l(\lambda, \tilde{\xi}) = i\xi_1(\mathrm{mod} 2\pi i/X)$ for some l. Setting $\mu = i\xi_1$ in (3.6), we obtain the dispersion relation $(-B^\xi - iA^\xi - \lambda I) = 0$, where $A^\xi = \sum_j A^j\xi_j$ and $B^\xi = \sum_{j,k} B^{jk}\xi_k\xi_j$.

4 WKB expansion and the low-frequency limit

We now recall the results of [OZ3] describing low-frequency spectral behavior. As a consequence of (H0), (H2), there is a smooth $n+d$ dimensional manifold $\mathcal{P}$ of periodic solutions $\dot{u}$ in the vicinity of $\bar{u}$, where d is the spatial dimension. On this set, one may obtain, rescaling by $(x,t) \to (\epsilon x, \epsilon t)$ and carrying out a formal WKB expansion as $\epsilon \to 0$ a closed system of $n+d$ averaged, or homogenized, equations

$$\begin{aligned} &\partial_t M(\dot{u}) + \sum_j \partial_{x_j}(F^j(\dot{u})) = 0, \\ &\partial_t(\Omega N(\dot{u})) + \nabla_x(\Omega S(\dot{u})) = 0 \end{aligned} \tag{4.1}$$

in the $(n+d)$-dimensional unknown $\dot{u}$, expected to correspond to large time-space behavior, with an additional constraint

$$\text{curl } (\Omega N) \equiv 0 \tag{4.2}$$

coming from the derivation of the formal expansion: specifically, the assumption that ΩN represent the gradient $\nabla_x \phi$ of a certain phase function $\phi(x,t)$. Here, Ω, M, etc. are defined as in (2.5); see [OZ3] for details.

The long-time behavior of perturbations of $\bar{u}$ can thus be studied formally by considering the linearized equations of (4.1) about the constant solution $\dot{u}(x,t) \equiv u^0$, $u^0 \sim \bar{u}$, yielding the homogeneous degree $n+d$ linearized dispersion relation

$$\hat{\Delta}(\xi,\lambda) := \det\left(\lambda \frac{\partial(M,\Omega N)}{\partial \dot{u}}(\dot{\bar{u}}) + \sum_j i\xi_j \frac{\partial(F^j, S\Omega e_j)}{\partial \dot{u}}(\dot{\bar{u}})\right) = 0. \tag{4.3}$$

Alternatively, it may be studied rigorously through low-frequency expansion of the Evans function $D(\xi,\lambda)$, $\xi \in \mathbb{R}^d$, $\lambda \in \mathbb{C}$, which yields after some standard but somewhat involved manipulations

$$D(\xi,\lambda) = \Delta_1(\xi,\lambda) + \mathcal{O}(|\xi,\lambda|^{n+2}), \tag{4.4}$$

where Δ_1 is a homogeneous degree $n+1$ polynomial expressed as the determinant of a rather complicated $2n \times 2n$ matrix in (ξ, λ). The zero set $(\xi, \lambda(\xi))$ of Δ_1 thus determines the linearized dispersion relation for (2.1), with $\lambda(\xi)$ running over the tangent cone at $\xi = 0$ to the surface of low-frequency spectrum of L as ξ runs over $\mathbb{R}^d$.

Our main result in [OZ3] was the following proposition relating these two expansions, generalizing the result of [Se1] in the one-dimensional case. Define

$$\Delta(\xi,\lambda) := \lambda^{1-d}\hat{\Delta}(\xi,\lambda), \tag{4.5}$$

where $\hat{\Delta}$ is defined as in (4.3).

Proposition 4.1. *[OZ3] Under assumptions (H0)–(H3), $\Delta_1 = \Gamma_0 \Delta$, i.e.,*

$$D(\xi,\lambda) = \Gamma_0 \Delta(\xi,\lambda) + \mathcal{O}(|\xi,\lambda|^{n+2}) \tag{4.6}$$

$\Gamma_0 \neq 0$ constant, for $|\xi,\lambda|$ sufficiently small.

That is, up to an additional factor of λ^{d-1} (corresponding to spurious modes not satisfying constraint (4.2; see [OZ3] for further discussion) the dispersion relation (4.3) for the averaged system (4.1) indeed describes the low-frequency limit of the exact linearized dispersion relation $D(\xi, \lambda) = 0$.

Corollary 4.1. *[OZ3] Assuming (H0)–(H3) and the nondegeneracy condition*

$$\det \left(\frac{\partial(M, \Omega N)}{\partial \dot{u}}(\dot{\bar{u}}) \right) \neq 0, \tag{4.7}$$

then for λ, ξ sufficiently small, the zero-set of $D(\cdot, \cdot)$, corresponding to spectra of L, consists of $n+1$ characteristic surfaces:

$$\lambda_j(\xi) = -ia_j(\xi) + o(\xi), \quad j = 1, \ldots, n+1, \tag{4.8}$$

where $a_j(\xi)$ denote the homogeneous, degree one eigenvalues of

$$\mathcal{A}(\xi) := \sum_j \xi_j \frac{\partial(F^j, S\Omega e_j)}{\partial(M, \Omega N)}, \tag{4.9}$$

excluding $(d-1)$ identically zero eigenvalues associated with modes not satisfying (4.2).

Corollary 4.2. *[OZ3] Assuming (H0)–(H3) and the nondegeneracy condition (4.7), a necessary condition for low-frequency spectral stability of $\bar{u}$, defined as $Re\, \lambda \leq 0$ for $D(\xi, \lambda) = 0$, $\xi \in \mathbb{R}^d$, and $|\xi, \lambda|$ sufficiently small, is that the averaged system (4.1) be "weakly hyperbolic" in the sense that the eigenvalues of $\mathcal{A}(\xi)$ are real for all $\xi \in \mathbb{R}^d$.*

A consequence of Corollary 4.1 is that $\lambda_j(\xi)$ are differentiable in $|\xi|$ at the origin for fixed angle $\hat{\xi}$, but in general have a conical singularity in ξ at $\xi = 0$, since the eigenvalues $a_j(\xi)$ of first-order system (4.1) are homogeneous degree one but typically not linear. The low-frequency expansion of Corollary 4.1 substitutes in our analysis for the usual spectral perturbation analysis by formal series expansion/Fredholm alternative in the standard case (as in [S1,S2,S3]) that $\lambda_j(\xi)$ vary smoothly in ξ.

Remark 4.1. The low-frequency stability condition of Corollary 4.2 has been verified numerically for the example of isentropic van der Waals gas dynamics [O], for which periodic solutions are known to appear. On the other hand, it was shown in [OZ1] that high-frequency instabilities appear for these waves, so they are not in the end stable.

5 Bloch–Fourier decomposition and the spectral stability conditions

Following [G,S1,S2,S3], we define the family of operators

$$L_\xi = \mathrm{e}^{-i\xi_1 x_1} L_{\tilde{\xi}} \mathrm{e}^{i\xi_1 x_1} \tag{5.1}$$

operating on the class of L^2 periodic functions on $[0, X]$; the (L^2) spectrum of $L_{\tilde{\xi}}$ is equal to the union of the spectra of all L_ξ with ξ_1 real with associated eigenfunctions

$$w(x_1, \tilde{\xi}, \lambda) := \mathrm{e}^{i\xi_1 x_1} q(x_1, \xi_1, \tilde{\xi}, \lambda), \tag{5.2}$$

where q, periodic, is an eigenfunction of L_ξ. By continuity of spectrum, and discreteness of the spectrum of the elliptic operators L_ξ on the compact domain $[0, X]$, we have that the spectra of L_ξ may be described as the union of countably many continuous surfaces $\lambda_j(\xi)$.

The spectrum of each L_ξ may alternatively be characterized as the zero set for fixed ξ of the periodic Evans function $D(\xi_1, \tilde{\xi}, \lambda)$; see [G,OZ3]. More, a fundamental result of Gardner [G] is that *the order of vanishing of the Evans function in λ is equal to the multiplicity of λ as an eigenvalue of L_ξ*. Thus, we have a description of the eigenstructure of L_ξ through Corollary 4.1 for λ, ξ sufficiently small in terms of the characteristics of the first-order hyperbolic system (4.1).

Without loss of generality taking $X = 1$, recall now the *Bloch–Fourier representation*

$$u(x) = \left(\frac{1}{2\pi}\right)^d \int_{-\pi}^{\pi} \int_{\mathbb{R}^{d-1}} e^{i\xi\cdot x} \hat{u}(\xi, x_1) d\xi_1 \, d\tilde{\xi} \tag{5.3}$$

of an L^2 function u, where $\hat{u}(\xi, x_1) := \sum_k e^{2\pi i k x_1} \hat{u}(\xi_1 + 2\pi k, \tilde{\xi})$ are periodic functions of period $X = 1$, $\hat{u}(\tilde{\xi})$ denoting with slight abuse of notation the Fourier transform of u in the full variable x. By Parseval's identity, the Bloch–Fourier transform $u(x) \to \hat{u}(\xi, x_1)$ is an isometry in L^2:

$$\|u\|_{L^2(x)} = \|\hat{u}\|_{L^2(\xi; L^2(x_1))}, \tag{5.4}$$

where $L^2(x_1)$ is taken on $[0, 1]$ and $L^2(\xi)$ on $[-\pi, \pi] \times \mathbb{R}^{d-1}$. Moreover, it diagonalizes the periodic-coefficient operator L, yielding the *inverse Bloch–Fourier transform representation*

$$e^{Lt} u_0 = \left(\frac{1}{2\pi}\right)^d \int_{-\pi}^{\pi} \int_{\mathbb{R}^{d-1}} e^{i\xi\cdot x} e^{L_\xi t} \hat{u}_0(\xi, x_1) d\xi_1 \, d\tilde{\xi} \tag{5.5}$$

relating behavior of the linearized system to that of the diagonal operators L_ξ.

Along with the assumptions (H0)–(H2), we assume the *strong spectral stability* conditions:

(D1) $\sigma(L) \subset \{\mathrm{Re}\lambda < 0\} \cup \{0\}$.

(D2) $\mathrm{Re}\sigma(L_\xi) \le -\theta|\xi|^2$, $\theta > 0$, for $\xi \in \mathbb{R}^d$ and $|\xi|$ sufficiently small.

By Corollary 4.1, either of (D1), (D2) implies that the eigenvalues $a_j(\xi)$ of (4.9) are real. We make the further nondegeneracy hypothesis:

(H3) The values $a_j(\xi)$ are distinct.

Conditions (D1)–(D2) are exactly the spectral assumptions of [S1,S2,S3], corresponding to "dissipativity" of the large-time behavior of the linearized system. Condition (H3) corresponds to strict hyperbolicity of the averaged system (4.1); presumably it could be removed with further effort/alternative hypotheses.

6 Linearized stability estimates

By standard spectral perturbation theory [K], the total eigenprojection $P(\xi)$ onto the eigenspace of L_ξ associated with the eigenvalues $\lambda_j(\xi)$, $j = 1, \dots, n+1$ described in Corollary 4.1 is well-defined and analytic in ξ for ξ sufficiently small, since these (by discreteness of the spectra of L_ξ) are separated at $\xi = 0$ from the rest of the spectrum of L_0. Introducing a smooth cutoff function $\phi(\xi)$ that is identically one for $|\xi| \le \epsilon$ and identically zero for $|\xi| \ge 2\epsilon$, $\epsilon > 0$ sufficiently small, we split the solution operator $S(t) := e^{Lt}$ into low- and high-frequency parts

$$S^I(t)u_0 := \Big(\frac{1}{2\pi}\Big)^d \int_{-\pi}^{\pi} \int_{\mathbb{R}^{d-1}} e^{i\xi\cdot x}\phi(\xi)P(\xi)e^{L_\xi t}\hat{u}_0(\xi, x_1)d\xi_1\, d\tilde{\xi} \tag{6.1}$$

and

$$S^{II}(t)u_0 := \Big(\frac{1}{2\pi}\Big)^d \int_{-\pi}^{\pi} \int_{\mathbb{R}^{d-1}} e^{i\xi\cdot x}(I - \phi P(\xi))e^{L_\xi t}\hat{u}_0(\xi, x_1)d\xi_1\, d\tilde{\xi}. \tag{6.2}$$

6.1 High-frequency bounds

By standard sectorial bounds [He,Pa] and spectral separation of $\lambda_j(\xi)$ from the remaining spectra of L_ξ, we have trivially the exponential decay bounds

$$\|e^{L_\xi}(I - \phi P(\xi))f\|_{L^2([0,X])} \le Ce^{-\theta t}\|f\|_{L^2([0,X])}, \tag{6.3}$$

$$\|e^{L_\xi}(I - \phi P(\xi))\partial_x f\|_{L^2([0,X])} \le Ct^{-\frac{1}{2}}e^{-\theta t}\|f\|_{L^2([0,X])}, \tag{6.4}$$

$$\|\partial_x e^{L_\xi}(I - \phi P(\xi))f\|_{L^2([0,X])} \le Ct^{-\frac{1}{2}}e^{-\theta t}\|f\|_{L^2([0,X])}, \tag{6.5}$$

for θ, $C > 0$. Together with (5.4), these give immediately the following estimates.

Proposition 6.1. *Under assumptions (H0)–(H3) and (D1)–(D2), for some θ, $C > 0$, and all $t > 0$, $2 \le p \le \infty$,*

$$\|S^{II}(t)f\|_{L^2(x)} \le Ce^{-\theta t}\|f\|_{L^2(x)}, \tag{6.6}$$

$$\|\partial_x S^{II}(t)f\|_{L^2(x)},\ \|S^{II}(t)\partial_x f\|_{L^2(x)} \le Ct^{-\frac{1}{2}}e^{-\theta t}\|f\|_{L^2(x)}), \tag{6.7}$$

$$\|S^{II}(t)f\|_{L^p(x)} \le Ct^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{p})}e^{-\theta t}\|f\|_{L^2(x)}. \tag{6.8}$$

Proof. The first two inequalities follow immediately by (5.4). The third follows for $p = \infty$ by Sobolev embedding from

$$\|S^{II}(t)f\|_{L^p(\tilde{x};L^2(x_1))} \le Ct^{-\frac{d-1}{2}(\frac{1}{2}-\frac{1}{p})}e^{-\theta t}\|f\|_{L^2([0,X])}$$

and

$$\|\partial_{x_1} S^{II}(t)f\|_{L^p(\tilde{x};L^2(x_1))} \le Ct^{-\frac{d-1}{2}(\frac{1}{2}-\frac{1}{p})-\frac{2}{2}}e^{-\theta t}\|f\|_{L^2([0,X])},$$

which follow by an application of (5.4) in the x_1 variable and the Hausdorff–Young inequality $\|f\|_{L^\infty(\tilde{x})} \le \|\hat{f}\|_{L^1(\tilde{\xi})}$ in the variable $\tilde{x}$. The result for general $2 \le p \le \infty$ then follows by L^p interpolation. ∎

6.2 Low-frequency bounds

Denote by

$$G^I(x,t;y) := S^I(t)\delta_y(x) \tag{6.9}$$

the Green kernel associated with S^I, and

$$[G^I_\xi(x_1,t;y_1)] := \phi(\xi)P(\xi)e^{L_\xi t}[\delta_{y_1}(x_1)] \tag{6.10}$$

the corresponding kernel appearing within the Bloch–Fourier representation of G^I, where the brackets on $[G_\xi]$ and $[\delta_y]$ denote the periodic extensions of these functions onto the whole line. Then, we have the following descriptions of G^I, $[G^I_\xi]$, deriving from the Evans function analysis of Corollary 4.1.

Proposition 6.2. *Under assumptions (H0)–(H3) and (D1)–(D2),*

$$[G^I_\xi(x_1,t;y_1)] = \phi(\xi)\sum_{j=1}^{n+1} e^{\lambda_j(\xi)t}q_j(\xi,x_1)\tilde q_j(\xi,y_1)^*, \tag{6.11}$$

$$\begin{aligned} G^I(x,t;y) &= \Big(\frac{1}{2\pi}\Big)^d\int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)}[G^I_\xi(x_1,t;y_1)]d\xi \\ &= \Big(\frac{1}{2\pi}\Big)^d\int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)}\phi(\xi)\sum_{j=1}^{n+1} e^{\lambda_j(\xi)t}q_j(\xi,x_1)\tilde q_j(\xi,y_1)^*d\xi, \end{aligned} \tag{6.12}$$

where $*$ denotes matrix adjoint, or complex conjugate transpose, $q_j(\xi,\cdot)$ and $\tilde q_j(\xi,\cdot)$ are right and left eigenfunctions of L_ξ associated with eigenvalues $\lambda_j(\xi)$ defined in Corollary 4.1, normalized so that $\langle \tilde q_j, q_j\rangle \equiv 1$, where $\lambda_j/|\xi|$ is a smooth function of $|\xi|$ and $\hat\xi := \xi/|\xi|$ and q_j and $\tilde q_j$ are smooth functions of $|\xi|$, $\hat\xi := \xi/|\xi|$, and x_1 or y_1, with $\Re\lambda_j(\xi) \le -\theta|\xi|^2$.

Proof. Smooth dependence of λ_j and of q, $\tilde q$ as functions in $L^2[0,X]$ follow from standard spectral perturbation theory [K] using the fact that λ_j split to first order in $|\xi|$ as ξ is varied along rays through the origin, and that L_ξ varies smoothly with angle $\hat\xi$. Smoothness of q_j, $\tilde q_j$ in x_1, y_1 then follow from the fact that they satisfy the eigenvalue equation for L_ξ, which has smooth, periodic coefficients. Likewise, (6.11) is immediate from the spectral decomposition of elliptic operators on finite domains. Substituting (6.9) into (6.1) and computing

$$\widehat{\delta_y}(\xi,x_1) = \sum_k e^{2\pi ikx_1}\widehat{\delta_y}(\xi+2\pi ke_1) = \sum_k e^{2\pi ikx_1}e^{-i\xi\cdot y-2\pi iky_1} = e^{-i\xi\cdot y}[\delta_y(x)], \tag{6.13}$$

where the second and third equalities follow from the fact that the Fourier transform either continuous or discrete of the delta-function is unity, we obtain

$$\begin{aligned} G^I(x,t;y) &= \Big(\frac{1}{2\pi}\Big)^d\int_{-\pi}^{\pi}\int_{\mathbb{R}^{d-1}} e^{i\xi\cdot x}\phi P(\xi)e^{L_\xi t}\widehat{\delta_y}(\xi,x_1)d\xi \\ &= \Big(\frac{1}{2\pi}\Big)^d\int_{-\pi}^{\pi}\int_{\mathbb{R}^{d-1}} e^{i\xi\cdot(x-y)}\phi P(\xi)e^{L_\xi t}[\delta_y(x)]d\xi, \end{aligned} \tag{6.14}$$

yielding (6.12) by (6.10) and the fact that ϕ is supported on $[-\pi,\pi]$. ∎

Proposition 6.3. *Under assumptions (H0)-(H3) and (D1)-(D3),*

$$\sup_y \|G^I(\cdot,t,;y)\|_{L^p(x)},\ \sup_y \|\partial_{x,y}G^I(\cdot,t,;y)\|_{L^p(x)} \le C(1+t)^{-\frac{d}{2}(1-\frac{1}{p}} \tag{6.15}$$

for all $2 \le p \le \infty$, $t \ge 0$, *where* $C > 0$ *is independent of* p.

Proof. From representation (6.11)(ii) and $\Re\lambda_j(\xi) \le -\theta|\xi|^2$, we obtain by the triangle inequality

$$\|G^I\|_{L^\infty(x,y)} \le C\|e^{-\theta|\xi|^2 t}\phi(\xi)\|_{L^1(\xi)} \le C(1+t)^{-\frac{d}{2}}, \tag{6.16}$$

verifying the bounds for $p = \infty$. Derivative bounds follow similarly, since derivatives falling on q_j or $\tilde{q}_j$ are harmless, whereas derivatives falling on $e^{i\xi\cdot(x-y)}$ bring down a factor of ξ, again harmless because of the cutoff function ϕ.

To obtain bounds for $p = 2$, we note that (6.11(ii) may be viewed itself as a Bloch–Fourier decomposition with respect to variable $z := x-y$, with y appearing as a parameter. Recalling (5.4), we may thus estimate

$$\sup_y \|G^I(x,t;y)\|_{L^2(x)} = \sum_j \sup_y \|\phi(\xi)e^{-\lambda_j(\xi)t}q_j(\cdot,z_1)\tilde{q}_j^*(\cdot,y_1)\|_{L^2(\xi;L^2(z_1\in[0,X]))} \tag{6.17}$$

$$\le C\sum_j \sup_y \|\phi(\xi)e^{-\theta|\xi|^2 t}\|_{L^2(\xi)}\|q_j\|_{L^2(0,X)}\|\tilde{q}_j\|_{L^\infty(0,X)} \tag{6.18}$$

$$\le C(1+t)^{-\frac{d}{4}}, \tag{6.19}$$

where we have used in a crucial way the boundedness of $\tilde{q}_j$; derivative bounds follow similarly.

Finally, bounds for $2 \le p \le \infty$ follow by L^p-interpolation. ■

Remark 6.1. In obtaining the key L^2-estimate, we have used in an essential way the periodic structure of q_j, $\tilde{q}_j$. For, viewing G^I as a general pseudodifferential expression rather than a Bloch–Fourier decomposition, we find that the smoothness of q_j, $\tilde{q}_j$ is not sufficient to apply standard $L^2 \to L^2$ bounds of Hörmander, which require blowup in ξ derivatives at *less than* the critical rate $|\xi|^{-1}$ found here; see, e.g., [H] for further discussion.

Remark 6.2. Computation (6.13) applied to the full solution formula (5.5) yields the fundamental relation

$$G(x,t;y) = \Big(\frac{1}{2\pi}\Big)^d \int_{-\pi}^{\pi}\int_{\mathbb{R}^{d-1}} e^{i\xi\cdot(x-y)}[G_\xi(x_1,t;y_1)]d\xi \tag{6.20}$$

which, provided $\sigma(L_\xi)$ is semisimple, yields the simple formula

$$G(x,t;y) = \Big(\frac{1}{2\pi}\Big)^d \int_{-\pi}^{\pi}\int_{\mathbb{R}^{d-1}} e^{i\xi\cdot(x-y)}\sum_j e^{\lambda_j(\xi)t}q_j(\xi,x_1)\tilde{q}_j(\xi,y_1)^* d\xi$$

resembling that of the constant-coefficient case, where λ_j runs through the spectrum of L_ξ. Relation (6.20) underlies both the present analysis and the technically rather different approach of [OZ2], with the basic idea in both cases being to separate off the principal part of the series involving small $\lambda_j(\xi)$ and estimate the remainder as a faster-decaying residual.

Corollary 6.1. *Under assumptions (H0)–(H3) and (D1)–(D2), for all $p \geq 2$, $t \geq 0$,*

$$|S^I(t)f\|_{L^p},\ \|\partial_x S^I(t)f\|_{L^p},\ \|S^I(t)\partial_x f\|_{L^p} \leq C(1+t)^{-\frac{d}{2}(1-\frac{1}{p})}\|f\|_{L^1}. \tag{6.21}$$

Proof. Immediate, from (6.15) and the triangle inequality, as, for example,

$$\|S^I(t)f(\cdot)\|_{L^p} = \Big\| \int_{\mathbb{R}^d} G^I(x,t;y)f(y)dy\Big\|_{L^p(x)} \leq \int_{\mathbb{R}^d} \sup_y \|G^I(\cdot,t;y)\|_{L^p}|f(y)|dy.$$

∎

Additional estimates. For general interest, we state also some easy-to-obtain generalizations, even though they are not needed in the analysis. From boundedness of the spectral projections $P_j(\xi) = q_j\langle \tilde{q}_j, \cdot\rangle$ in $L^2[0,X]$ and their derivatives, another consequence of first-order splitting of eigenvalues $\lambda_j(\xi)$ at the origin, we obtain boundedness of $\phi(\xi)P(\xi)e^{L_\xi t}$ and thus, by (5.4), the global bounds

$$\|S^I(t)f\|_{L^2(x)},\ \|\partial_x S^I(t)f\|_{L^2(x)},\ \|S^I(t)\partial_x f\|_{L^2(x)} \leq C\|f\|_{L^2(x)} \quad \text{for all } t \geq 0. \tag{6.22}$$

By Riesz–Thorin interpolation between (6.22) and (6.21), we obtain the following, apparently sharp bounds between various L^q and L^p.

Corollary 6.2. *Assuming (H0)–(H3) and (D1)–(D2), for all $1 \leq q \leq 2$, $p \geq 2$, $t \geq 0$,*

$$\|S^I(t)f\|_{L^p},\ \|\partial_x S^I(t)f\|_{L^p},\ \|S^I(t)\partial_x f\|_{L^p} \leq C(1+t)^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}\|f\|_{L^q}. \tag{6.23}$$

6.3 Short-time bounds

Finally, we recall the following short-time bounds on the full solution operator $S(t) = e^{Lt}$, following from standard semigroup theory [Pa] for the second-order elliptic operator L.

Proposition 6.4. *Assuming (H0)–(H3) and (D1)–(D2), for $1 \leq p \leq \infty$, $0 \leq t \leq 1$,*

$$\|S(t)f\|_{L^p} \leq C\|f\|_{L^p}, \tag{6.24}$$

$$\|\partial_x S(t)f\|_{L^p},\ \|S(t)\partial_x f\|_{L^p} \leq Ct^{-\frac{1}{2}}\|f\|_{L^p}. \tag{6.25}$$

6.4 Linearized stability in dimensions $d \geq 1$

Theorem 1. *Assuming (H0)-(H3), spectral stability (D1)-(D2), implies $L^1 \cap L^p \to L^p$ asymptotic stability of the linear equation (2.1), for all $p \geq 2$ and dimensions $d \geq 1$, with*

$$\|S(t)u_0\|_{L^p} \leq C(1+t)^{-\frac{d}{2}(1-\frac{1}{p})}\|u_0\|_{L^1 \cap L^p}, \tag{6.26}$$

$$\|S(t)\partial_x u_0\|_{L^p} \leq Ct^{-\frac{1}{2}}(1+t)^{\frac{1}{2}-\frac{d}{2}(1-\frac{1}{p})}\|u_0\|_{L^1 \cap L^p} \quad \text{for all } t \geq 0. \tag{6.27}$$

Proof. Immediate, from (6.6), (6.21), and (6.24). ∎

7 Nonlinear stability in dimensions $d \geq 3$

Define now the perturbation variable $v := u - \bar{u}$ for u a solution of (2.1).

Proposition 7.1. *Assuming (H0)-(H3), let $v_0 \in H^k$, $k \geq [d/2]+1$ as in (H0), and suppose that for $0 \leq t \leq T$, the H^k norm of v remains bounded by a sufficiently small constant. There are then constants $\theta_{1,2} > 0$ so that, for all $0 \leq t \leq T$,*

$$\|v(t)\|_{H^k}^2 \leq C\mathrm{e}^{-\theta_1 t}\|v(0)\|_{H^k}^2 + C\int_0^t \mathrm{e}^{-\theta_2(t-s)}|v|_{L^2}^2(s)\,ds. \tag{7.1}$$

Proof. Subtracting the equations for u and $\bar{u}$, we may write the nonlinear perturbation equation as

$$\begin{aligned} v_t + \sum_j (df_j(\bar{u})v)_{x_j} - \sum_{j,k}(B_{jk}(u)v_{x_j})_{x_k} &= \sum_{j,k}((B_{jk}(\bar{u}+v) - B_{jk}(\bar{u}))\bar{u}_{x_j})_{x_k} \\ &\quad - \sum_j (f_j(\bar{u}+v) - f_j(\bar{u}) - df_j(\bar{u})v)_{x_j}. \end{aligned} \tag{7.2}$$

In the uniformly elliptic case

$$\Re \sum_{jk} B_{jk}\nu_j\nu_k \geq \theta|\nu|^2,\ \theta > 0$$

we may take the L^2 inner product in x of $\sum_{k=0}^k \partial_x^{2k} v$ against (7.2), integrate by parts, apply Gärding's inequality, and rearrange the resulting terms, to arrive at the inequality

$$\partial_t\|v\|_{H^k}^2(t) \leq -\theta\|\partial_x^{k+1}v\|_{L^2}^2 + C\|v\|_{H^k}^2,$$

where $\theta > 0$, for some sufficiently large $C > 0$, so long as $\|v\|_{H^k}$ remains bounded. Using the Sobolev interpolation $\|v\|_{H^k}^2 \leq \tilde{C}^{-1}\|\partial_x^{k+1}v\|_{L^2}^2 + \tilde{C}\|v\|_{L^2}^2$ for $\tilde{C} > 0$ sufficiently large, we obtain $\partial_t\|v\|_{H^k}^2(t) \leq -\tilde{\theta}\|v\|_{H^k}^2 + C\|v\|_{L^2}^2$, from which (7.1) follows by Gronwall's inequality. In the general case, we perform an analogous pseudodifferential estimate using a frequency-dependent symmetrizer to obtain the same result. We omit the (standard) details. ∎

Theorem 2. *Assume (H0)-(H3). Then, spectral stability, (D1)-(D3), implies nonlinear asymptotic stability of $\bar u$ from $L^1 \cap H^k \to L^p$, $k \geq [d/2]+1$ as in (H0) and $p \geq 2$, for dimensions $d \geq 2$, with*

$$\|u(\cdot,t)-\bar u\|_{L^p} \leq C(1+t)^{-\frac{d}{2}(1-\frac{1}{p})}\|u_0-\bar u\|_{L^1\cap H^k}, \tag{7.3}$$

$$\|u(\cdot,t)-\bar u\|_{H^k} \leq C(1+t)^{-\frac{d}{4}}\|u_0-\bar u\|_{L^1\cap H^k} \quad \text{for all } t \geq 0. \tag{7.4}$$

Proof. Taylor expanding about $\bar u$, we obtain the alternative perturbation equation

$$v_t - Lv = \sum_j Q^j(v,\nabla v)_{x_j}, \tag{7.5}$$

where

$$Q^j(v,\nabla v) = \mathcal{O}(|v|^2 + |v||\nabla v|) \tag{7.6}$$

as long as $|v|$ remains bounded by some fixed constant. By Duhamel's principle, we have

$$v(\cdot,t) = S(t)v_0 + \int_0^t S(t-s)\sum_j \partial_{y_j} Q^j(\cdot,s)ds. \tag{7.7}$$

Case $L^2 \cap H^k$. Define now

$$\eta(t) := \sup_{0\leq s\leq t}\|v(\cdot,s)\|_{L^p}(1+s)^{\frac{d}{4}}. \tag{7.8}$$

By Proposition 7.1, $\|v\|_{H^2}$, hence $\|v_t\|_{L^2}$, remains small so long as η remains small, hence η remains continuous so long as it remains small. We first establish

$$\eta(t) \leq C(\eta_0 + \eta(t)^2), \quad \eta_0 := \|v_0\|_{L^1\cap H^k}, \tag{7.9}$$

from which it follows by continuous induction that $\eta(t) \leq 2C\eta_0$ for $t \geq 0$, if $\eta_0 < 1/4C$, yielding by (7.8) the result (7.3) for $p=2$. This in turn yields (7.4) by Proposition 7.1.

By Proposition 7.1, (7.8), and Sobolev embedding,

$$|Q^j(v,\nabla v)(\cdot,t)|_{L^1\cap L^p} \leq (|\nabla v|_{L_2} + |v|_{L^2})|v|_{L^2\cap L^\infty} \leq C\eta(t)^2(1+t)^{-\frac{d}{2}} \tag{7.10}$$

for all $2 \leq p \leq \infty$, in particular $p=2$. Substituting into (7.7) and using (6.26), (6.27), we thus obtain

$$\begin{aligned} |v(\cdot,t)|_{L^2} &\leq C\eta_0(1+t)^{-\frac{d}{4}} \\ &\quad + C\eta(t)^2\int_0^t (1+t-s)^{\frac{1}{2}-\frac{d}{4}}(t-s)^{-\frac{1}{2}}(1+s)^{-\frac{d}{2}} \\ &\leq C(\eta_0+\eta(t)^2)(1+t)^{\max\{-\frac{d}{4},1-\frac{3d}{4}\}} \end{aligned} \tag{7.11}$$

so long as $(1+s)^{-\frac{d}{2}}$ is integrable, i.e., for $d \geq 3$. Noting that $1-\frac{3d}{4} \leq -\frac{d}{4}$ for $d \geq 2$, we obtain (7.9) as claimed. This completes the proof of (7.3)–(7.4) for $p=2$.

Case L^p, $2 \le p \le \infty$. Substituting again into (7.7) and using (6.26), (6.27), we obtain

$$
\begin{aligned}
|v(\cdot,t)|_{L^2} &\le C\eta_0(1+t)^{-\frac{d}{2}(1-\frac{1}{p})} \\
&\quad + C\eta(t)^2 \int_0^t (1+t-s)^{\frac{1}{2}-\frac{d}{2}}(t-s)^{-\frac{1}{2}(1-\frac{1}{p})}(1+s)^{-\frac{d}{2}} \\
&\le C_2\eta_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}
\end{aligned}
\tag{7.12}
$$

for $d \ge 3$. This completes the proof of (7.3) for $2 \le p \le \infty$. ∎

Remark 7.1. In the critical dimension $d = 2$, our stability argument barely fails, due to the appearance of a $\log t$ term coming from the expression $\int (1+s)^{-\frac{d}{2}} ds$.

8 Asymptotic behavior in dimensions $d \ge 3$

8.1 Second-order approximation

Expressing ξ in polar coordinates $(r, \hat\xi)$, where $\xi = r\hat\xi$, $r = |\xi|$, $\hat\xi = \xi/|\xi|$, and expanding

$$\lambda_j(r, \hat\xi) = a_j(\hat\xi) r + b_j(\hat\xi) r^2 + O(r^3),$$

define now

$$\lambda_j^*(\xi) := a_j(\hat\xi) r + b_j(\hat\xi) r^2, \quad j = 1, \dots, n+1, \tag{8.1}$$

$$[G_\xi^\dagger(x_1, t; y_1)] := \phi(\xi) \sum_{j=1}^{n+1} e^{\lambda_j^\dagger(\xi) t} q_j((0,\hat\xi), x_1)\tilde q_j((0,\hat\xi), y_1)^*, \tag{8.2}$$

$$G^\dagger(x,t;y) := \Big(\frac{1}{2\pi}\Big)^d \int_{\mathbb{R}^d} e^{i\xi\cdot(x-y)}[G_\xi^\dagger(x_1,t;y_1)]d\xi, \tag{8.3}$$

noting that $q((r,\hat\xi), x_1)$ and $\tilde q((r,\hat\xi), y_1)$ by Proposition 6.2 are smooth in all arguments. Define likewise

$$v^\dagger(x,t) := \int_{\mathbb{R}^d} G^\dagger(x,t;y)v_0(y)dy = \int_{\mathbb{R}^d} G^\dagger(x,t;y)(u_0-\bar u)(y)dy. \tag{8.4}$$

Proposition 8.1. *Assuming (H0)-(H3) and (D1)-(D3), for all $p \ge 2$, $d \ge 3$, $t \ge 0$,*

$$\|u(\cdot,t) - \bar u - v^\dagger(\cdot,t)\|_{L^p} \le C(1+t)^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}}\|u_0-\bar u\|_{L^1\cap H^k}, \tag{8.5}$$

$$\|u(\cdot,t) - \bar u - v^\dagger(\cdot,t)\|_{H^k} \le C(1+t)^{-\frac{d}{4}-\frac{1}{2}}\|u_0-\bar u\|_{L^1\cap H^k}. \tag{8.6}$$

Proof. By smoothness of q_j, $\tilde q_j$ in angle $\hat\xi$,

$$|q_j((r,\hat\xi),x_1) - q_j((0,\hat\xi),x_1)|,\ |\tilde q_j((r,\hat\xi),x_1) - \tilde{q_j}((0,\hat\xi),x_1)| = O(|\xi|),$$

whence we easily obtain (8.5) and (8.6) by estimates like those used in the proof of Theorem 2. We omit the details; see, for example, [HoZ1] for similar computations. ∎

8.2 Reduction to constant-coefficients

Noting that $\mathrm{Span}\{q_j\}(0,\hat\xi)$ and $\mathrm{Span}\{\tilde q_j\}(0,\hat\xi)$ are independent of the angle $\hat\xi$, corresponding for each $\hat\xi$ to the right and left zero eigenspaces Σ_0 and $\tilde\Sigma_0$ of L_0, we may choose a fixed real (since L_0 has real coefficients and eigenvalue 0 is real) pair of dual bases π_j and $\tilde\pi_j$ of Σ_0 and $\tilde\Sigma_0$, $\langle\tilde\pi_j,\pi_k\rangle=\delta_j^k$, $j=1,\dots,n+1$, to obtain

$$q_j((0,\hat\xi),x_1)=\sum_k \alpha_{kj}(0,\hat\xi)\pi_k(x_1), \tag{8.7}$$

$$\tilde q_j((0,\hat\xi),y_1)=\sum_k \tilde\alpha_{kj}(0,\hat\xi)\tilde\pi_k(y_1), \tag{8.8}$$

for some smooth nonsingular matrix-valued functions α, $\tilde\alpha\in\mathbb{R}^{(n+1)\times(n+1)}$, with

$$\tilde\alpha^*\alpha=I_{n+1}.$$

Denoting by α_j, $\tilde\alpha_j$ the jth columns of α, $\tilde\alpha$, we obtain the Floquet-type factorization

$$G^\dagger(x,t;y)\;=\;\Pi(x_1)g^\dagger(x-y,t)\tilde\Pi(y_1)^*, \tag{8.9}$$

$$g^\dagger(x-y,t):=\Big(\frac{1}{2\pi}\Big)^d\int_{\mathbb{R}^d}e^{i\xi\cdot(x-y)}[g^\dagger_\xi(t)]d\xi, \tag{8.10}$$

$$[g^\dagger_\xi(t)]:=\phi(\xi)\sum_{j=1}^{n+1}e^{\lambda_j^\dagger(\xi)t}\alpha_j(0,\hat\xi)\tilde\alpha_j(0,\hat\xi)^*, \tag{8.11}$$

where $\Pi:=(\pi_1,\dots,\pi_{n+1})$, $\tilde\Pi:=(\tilde\pi_1,\dots,\tilde\pi_{n+1})$, and g is a constant-coefficient operator in the sense that it is invariant under spatial translations. Recall, by definition, that $\tilde\Pi^*\Pi=I_n$, so that $\tilde\Pi^*$ is something like a pseudoinverse of Π (not "the" pseudoinverse, however, except in the self-adjoint case $\Sigma_0=\tilde\Sigma_0$).

Indeed, we may factor a bit further, as

$$g^\dagger=W*K=K*W, \tag{8.12}$$

where

$$W(z,t):=\Big(\frac{1}{2\pi}\Big)^d\int_{\mathbb{R}^d}e^{i\xi\cdot z}\phi(\xi)\sum_{j=1}^{n+1}e^{a_j(\xi)t}\alpha_j(0,\hat\xi)\tilde\alpha_j(0,\hat\xi)^*d\xi \tag{8.13}$$

denotes the hyperbolic part of the solution operator and

$$K(z,t):=\Big(\frac{1}{2\pi}\Big)^d\int_{\mathbb{R}^d}e^{i\xi\cdot z}\phi(\xi)\sum_{j=1}^{n+1}e^{b_j(\xi)t}\alpha_j(0,\hat\xi)\tilde\alpha_j(0,\hat\xi)^*d\xi \tag{8.14}$$

denotes the diffusive part, with $b_j(\xi):=|\xi|^2b_j(\hat\xi)$. This gives a description of the constant-coefficient solution operator $s^\dagger(t)f:=g^\dagger(\cdot,t)*f$ as the composition $S^\dagger=S_W\dot S_K=S_K\dot S_W$ of commuting hyperbolic and parabolic solution operators $S_W(t)f:=W(\cdot,t)*f$, $S_K(t)f:=K(\cdot,t)*f$, and of the Green kernel $g^\dagger$ as a *linear convection–diffusion wave* $W*K=S_W(t)K$ as defined in [HoZ1] for general constant-coefficient systems.

8.3 Convergence to linear convection–diffusion wave

Define now $w(x,t) := \tilde{\Pi}^*(x_1)v^\dagger(x,t) \in \mathbb{R}^{n+1}$. Evidently,

$$v^\dagger(x,t) = \Pi(x_1)w(x,t), \quad w = g^\dagger * w_0, \quad w_0(x) := \tilde{\Pi}^*(x_1)v_0(x). \tag{8.15}$$

Denote by

$$W := \int_{\mathbb{R}^d} w_0(x)dx = \int_{\mathbb{R}^d} \tilde{\Pi}(x_1)^* v_0(x)dx \tag{8.16}$$

the total mass of w_0.

Theorem 3. *Assuming (H0)-(H3) and (D1)-(D3), for $d \geq 3$, $t \geq 1$,*

$$\|u(\cdot,t) - \bar{u} - \Pi g^\dagger(\cdot,t)W\|_{L^p} \leq C(1+t)^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}}(\|v_0\|_{L^1\cap H^k} + \|xw_0\|_{L^1}), \tag{8.17}$$
$$\|u(\cdot,t) - \bar{u} - \Pi g^\dagger(\cdot,t)W\|_{H^k} \leq C(1+t)^{-\frac{d}{4}-\frac{1}{2}}(\|v_0\|_{L^1\cap H^k} + \|xw_0\|_{L^1}), \tag{8.18}$$

where $W \in \mathbb{R}^{n+1}$ is the constant mass vector defined in (8.16). Moreover,

$$C_1(1+t)^{-\frac{d}{4}} \leq \|g^\dagger(\cdot,t)\|_{L^2} \leq C_2(1+t)^{-\frac{d}{4}}, \tag{8.19}$$

so that in general $\|u(\cdot,t) - \bar{u} - \Pi g^\dagger(\cdot,t)W\|_{L^2} << \|\Pi g^\dagger(\cdot,t)W\|_{L^2}$, or $u(\cdot,t) - \bar{u} \sim \Pi g^\dagger W$.

Proof. By Proposition 8.1, (8.15), and boundedness of Π and derivatives, it is sufficient to show that

$$\|g^\dagger * w_0 - g^\dagger W\|_{L^p} \leq C(1+t)^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}}\|xw_0\|_{L^1},$$
$$\|g^\dagger * w_0 - g^\dagger W\|_{H^k} \leq C(1+t)^{-\frac{d}{4}-\frac{1}{2}}\|xw_0\|_{L^1},$$

or, by Hausdorff–Young's inequality and Parseval's identity, that

$$\|\widehat{g^\dagger}(\hat{w}_0(\xi) - W)\|_{L^q} \leq C(1+t)^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}}\|xw_0\|_{L^1}, \tag{8.20}$$
$$\|(1+|\xi|^k)\widehat{g^\dagger}(\hat{w}_0(\xi) - W)\|_{L^2(\xi)} \leq C(1+t)^{-\frac{d}{4}-\frac{1}{2}}\|xw_0\|_{L^1}, \tag{8.21}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Noting, by the Mean Value Theorem and Hausdorff–Young's inequality, that

$$\|\hat{w}_0(\xi) - W\|_{L^\infty(\xi)} = \|\hat{w}_0(\xi) - \hat{w}_0(0)\|_{L^\infty(\xi)} \leq |\xi|\|\partial_\xi \hat{w}_0\|_{L^\infty(\xi)} \leq |\xi|\|\partial_\xi \hat{w}_0\|_{L^\infty(\xi)} \leq |\xi|\|xw_0\|_{L^1},$$

we readily obtain (8.20)–(8.21) from

$$\|\xi\widehat{g^\dagger}\|_{L^q} \leq C(1+t)^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}} \quad \text{and} \quad \|\xi(1+|\xi|^k)\widehat{g^\dagger}\|_{L^2(\xi)} \leq C(1+t)^{-\frac{d}{4}-\frac{1}{2}},$$

as follow by direct computation from representation (8.10)–(8.11) as, likewise, does (8.19); see [HoZ1] for similar computations. ∎

9 Discussion and open problems

Theorem 3 shows that the L^2-asymptotic behavior of perturbations v with initial value v_0 possessing an L^1 first moment is given by a linear convection-diffusion wave

$$\Pi(x_1)g^\dagger(x,t)M = \Pi(x_1)W * K(x,t)M$$

with amplitude determined by the modulated mass $M = \int_{\mathbb{R}^d} \tilde{\Pi}^*(x_1)v_0(x)dx$, decaying in L^2 at the rate of a heat kernel (Gaussian). Due to the convective-diffusive structure of $g^\dagger$, this is essentially all we can say without further assumptions on the explicit structure of the hyperbolic system (4.1). For, geometric effects such as focusing or defocusing of characteristics can greatly affect the L^p norm of $W * K$ for norms $p > 2$, as discussed in [HoZ1] for the specific case of the wave equation. It might even be that for sufficiently large p a different part of the solution dominates behavior.

A brief consideration reveals that the zero eigenspace of L_0 spaned by the columns of Π consists of tangent directions along the manifold of possible periodic solutions nearby $\bar{u}$. Thus, our description (8.9) of lowest-order behavior as the product of Π and a solution $g^\dagger * (\tilde{\pi}v_0)$ of a diffusive regularization of a hyperbolic system corresponding to (4.1) can be viewed roughly as a linearized version of the formal description by WKB approximation, in which, to lowest order, the solution is approximated by

$$\bar{u}^{\zeta(x,t)}(\psi(x,t)), \tag{9.1}$$

where ζ indexes the manifold of nearby traveling waves and ψ is a scalar phase function with $\nabla_x\psi = N\Omega$; see [Se1,OZ3] for further discussion.

For lower dimensions $d = 1$ and 2 where decay of the linearized solution is slower, behavior is not expected to be dominated by its linear part, and indeed we have seen that the description of the solution as linear part plus error is too crude even to close a stability argument. A very interesting direction for further investigation would be to attempt to encode the lowest-order behavior at a nonlinear level using (9.1) or a slight modification, so as to eliminate the largest terms in the nonlinear residual and close the stability argument. See, for example, the argument used in [HoZ2] to obtain stability and behavior of scalar viscous shock fronts in the critical dimension $d = 2$, in which quite similar difficulties arise. See also the remarkable work of Schneider [S1,S2,S3] on stability of patterns in dimensions $d = 1, 2$, in which the nonlinear behavior encoded by modulation equations likewise plays a crucial role in the analysis by revealing unexpected cancellation needed to close the stability estimates.

We remark that our way of getting linearized L^2 estimates based on isometry properties is essentially different from, and somewhat simpler than, either the weighted norm approach of [S1,S2,S3] or the one-dimensional pointwise approach of [OZ1]. This more primitive approach allows us to treat cases, as here, for which the low-frequency dispersion relation is not smooth, as can be expected for general systems for which convection plays a role.

Finally, we recall that the analysis of [S1,S2,S3] concerns general multiply periodic waves, i.e., waves that are either periodic or else constant in each coordinate direction. It would be very interesting to consider whether such waves with two or more periods can arise as solutions of conservation laws, and if so, what would be the resulting behavior, even at a formal WKB level.

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